GAUGE THEORETIC APPROACH TO FLUID DYNAMICS: NON-MINIMAL MHD AND EXTENDED SPACE QUANTIZATION

Subir Ghosh

Physics and Applied Mathematics Unit,
Indian Statistical Institute,
203 B. T. Road, Calcutta 700035,
India.

Abstract:
The principle of local gauge invariance is introduced in the study of the Hamiltonian dynamics of a compressible inviscid fluid. In the first part of the paper, a novel gauging of the global symmetries is proposed which is both non-minimal and non-linear. The resulting MHD is analysed in detail. In the second part, the free fluid model is embedded in an extended space in the Batalin-Tyutin framework and the induced gauge theory is analysed. Relativistic generalizations are also discussed for both the cases.

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The principle of local gauge invariance has permeated through the present day theoretical physics. On the one hand, it has paved the way in formulating interacting theories between matter and abelian or non-abelian gauge fields, (*i.e.* photons or gluons respectively), in uncharted territories, whereas on the other hand, it has generated a successful scheme of quantizing a constrained system. In the present work, we will focus on all of the above aspects of the gauging process, the underlying matter theory being the Hamiltonian formulation of fluid dynamics [1], which, recently has grabbed a lot of attention from particle physicists due its close connection with high energy physics models, such as d-branes etc. [2]. We will be considering inviscid, isentropic and compressible fluid, whose dynamics is governed by the continuity equation and Euler equation,

$$\partial_t\rho + \partial_i(\rho v^i) = 0 \ , \quad \partial_t v_i + (v^j\partial_j)v_i = f_i, \tag{1}$$

where ρ and v_i denote the density and velocity fields respectively. We keep f_i arbitrary for the time being. The above equations of motion (1) are generated by

$$\partial_t\rho(x) = \{\rho(x), H\} \ , \partial_t v_i(x) = \{v_i(x), H\} \ , \quad H[\rho, v_i] = \int d^3y \mathcal{H}(y), \tag{2}$$

using the following Hamiltonian and the (non-canonical) Poisson Bracket (PB) algebra [3]

$$\mathcal{H} = \frac{1}{2}\rho v_i v_i + U \ , \quad \{v_i(x), \rho(y)\} = \frac{\partial}{\partial x^i}\delta(x-y) \ , \quad \{v_i(x), v_j(y)\} = -\frac{\partial_i v_j - \partial_j v_i}{\rho}\delta(x-y). \tag{3}$$

Note that we consider only those f_i in (1) which can be generated by some U.

However, a canonical Lagrangian formulation of the above is lacking due to the presence of the fluid helicity term,

$$C = \int d^3x(\epsilon_{ijk}v^i\partial_j v^k),$$

which, being a Casimir of the theory, creates an obstruction in the inversion of the symplectic matrix [4, 1]. To overcome this problem, albeit in the special case of helicity-less flows, Clebsch parametrization [5] of the velocity field v_i and its associated PB algebra are introduced,

$$v_i(x) \equiv \partial_i\theta(x) + \alpha(x)\partial_i\beta(x), \tag{4}$$

$$\{\theta(x), \alpha(y)\} = -\frac{\alpha}{\rho}\delta(x-y) \ , \ \{\beta(x), \alpha(y)\} = \frac{1}{\rho}\delta(x-y) \ , \ \{\theta(x), \rho(y)\} = \delta(x-y). \tag{5}$$

These are the only non-zero PBs. This parametrization renders the helicity variable to a surface term without any bulk contribution and obviously reproduces the previous equations of motion. The Lagrangian providing the correct symplectic structure [1] and equations of motion is

$$\mathcal{L} = \dot\theta\rho + \dot\beta\alpha\rho - (\frac{1}{2}\rho v_i v_i + U). \tag{6}$$

In our subsequent discussions, we will always use the Clebsch variables.

Recently it has been demonstrated [6], that the fluid system, even in presence of (particular types of) non-trivial interactions, supports infinite number of independent conservation laws and one can identify the conserved currents as Noether currents related to the corresponding global symmetry of the action. This observation has raised the question of integrability [7] of the fluid system. But in another vein, it is also natural to consider the promotion of the above global symmetries to local (gauge) invariances by suitably introducing gauge field interactions. This gives rise to the non-minimal magnetohydrodynamics, which is developed in Section II.

Next comes the quantization of the fluid system, where the naive interpretation of the PB algebra as the canonical commutation relations is problematic, since the algebra (5) is non-canonical and field dependent. This problem is solved by first enlarging the phase space [8] along with Second Class Constraints (SCC, [9]) which reduce the system to the original one. Subsequently we utilise the well studied [10] Batalin-Tyutin (BT) extension scheme [11] where extra variables are incorporated to convert the SCCs to First Class Constraints (FCC, [9]). The resulting gauge theory is subjected to quantisation since the PB algebra in the extended space is completely canonical. This is dealt with in Section III. For a simplified irrotational flow, BT extension is discussed in [12].

The BT construction of the fluid can also be useful in the classical context as the gauge freedom can be exploited to establish gauge equivalence between apparantly distinct fluid systems.

The paper ends with a conclusion and future prospects in Section IV.

Section II: Non-minimal MHD

We start by exhibiting the conservation laws and related Lagrangian symmetries of the fluid system. Time evolution of the variables in the free system (*i.e.* U=0 in (3)) are,

$$\{\alpha(x),\frac{1}{2}\int d^3y(\rho v_i v_i)\} = v_i\partial_i\alpha \;\; ; \;\; \dot{\beta}(x) = v_i\partial_i\beta;$$

$$\dot{\theta} = v_i(\partial_i\theta - \frac{v_i}{2}) \;\; ; \;\; \dot{\rho} = \partial_i(\rho v_i). \tag{7}$$

Using the above relations, one can easily check that [6]

$$\partial_t(A(\alpha)\rho) = \partial_i(A(\alpha)\rho v_i) \; ; \; \partial_t(B(\beta)\rho) = \partial_i(B(\beta)\rho v_i) \; ; \; \partial_t(C(\alpha,\beta)\rho) = \partial_i(C(\alpha,\beta)\rho v_i), \tag{8}$$

where A, B and C are smooth functions. The above constitute the three independent serieses of local current conservation equations. Clearly the conserved charges $A[\alpha] = \int(A(\alpha)\rho)$ etc. are trivially involutive [6],

$$\{A_1[\alpha], A_2[\alpha]\} = \{B_1[\beta], B_2[\beta]\} = 0, \tag{9}$$

whereas terms of different serieses, between themselves, do not commute (in the PB sense).

Coming to the invariances of the action, we notice [6] an interesting fact that the action possesses the symmetries in a restricted way that is the arbitrary functions $A(\alpha)$, $B(\beta)$ and $C(\alpha,\beta)$ have to be replaced by *powers* in the respective fields only. This yields the conervation laws in the form,

$$\partial_t(\alpha^n\rho/n) = \partial_i(\alpha^n\rho v_i/n), \tag{10}$$

$$\partial_t(\beta^n\rho) = \partial_i(\beta^n\rho v_i), \tag{11}$$

$$\partial_t(\alpha^n\beta^m\rho/m) = \partial_i(\alpha^n\beta^m\rho v_i/m), \tag{12}$$

where n and m can be any number. The following are the three sets of conserved charges:

$$I_n = \int d^3x(\alpha^n\rho/n) \; , \; J_n = \int d^3x(\beta^n\rho) \; , K_{(n,m)} = \int d^3x(\alpha^n\beta^m\rho/m). \tag{13}$$

Clearly

$$\{I_n, I_m\} = \{J_n, J_m\} = 0, \tag{14}$$

and the rest of the PBs between charges are non-vanishing. It is important to notice that the conservation laws in (10), (11) and (12) remain valid in the presence of an interaction terms U in the Hamiltonian of the forms $U[\alpha, \rho]$, $U[\beta, \rho]$ and $U(\alpha^n \beta^m, \rho)$ respectively.

The transformations that keeps the action invariant and yields the conservation laws are respectively,

$$\theta \rightarrow \theta - \frac{n-1}{n}\alpha^n \eta \; ; \; \beta \rightarrow \beta + \alpha^{n-1}\eta \; ; \; \delta\alpha = 0 \; , \quad n \neq 0, \tag{15}$$

$$\theta \rightarrow \theta + \beta^n \eta \; ; \; \alpha \rightarrow \alpha - n\beta^{n-1}\eta \; ; \; \delta\beta = 0 \; , \tag{16}$$

$$\theta \rightarrow \theta - \frac{n-1}{m}\alpha^n \beta^m \eta \; ; \; \alpha \rightarrow \alpha - \alpha^n \beta^{m-1}\eta \; ; \; \beta \rightarrow \beta + \frac{n}{m}\alpha^{n-1}\beta^m \eta \; , \quad m \neq 0, \tag{17}$$

where η is the infinitesimal global parameter. It can easily be checked that the conserved charges (13) act as the generators of the above transformations. Notice that for $n = 0$ in (11), the corresponding conservation law reduces to the mass conservation equation of (1). The above current expressions are reproduced from the Noether prescription

$$j^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_i}\delta\phi_i \;\; with \;\; \phi_i \equiv (\rho, \theta, \alpha, \beta) \;\; ; \;\; \partial^\mu j_\mu = 0.$$

Finally we are in a position to introduce the electromagnetic interactions in a gauge invariant manner. Notice that corresponding to each set of conservation laws, the Lagrangian has a global $U(1) \times U(1) \times ....$ symmetry for each value of n or m. We will discuss the first case (10) in detail. Taking the Lagrangian

$$\mathcal{L} = \rho(\dot{\theta} + \alpha\dot{\beta}) - \frac{\rho}{2}(\partial_i\theta + \alpha\partial_i\beta)^2 - U(\alpha, \rho),$$

we gauge it to the form

$$\mathcal{L} = \rho(\dot{\theta} + \alpha\dot{\beta} - e\frac{\alpha^n}{n}A_0) - \frac{\rho}{2}(\partial_i\theta + \alpha\partial_i\beta - e\frac{\alpha^n}{n}A_i)^2 - U(\alpha, \rho) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{18}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The action is invariant under the combined transformations

$$\delta\theta = -\frac{n-1}{n}\alpha^n \eta \; ; \; \delta\beta = \alpha^{n-1}\eta \;\; ; \delta\alpha = 0 \;\; ; \;\; \delta A_\mu = \frac{1}{e}\partial_\mu \eta \;\; , \;\; n \neq 0.$$

A straightforward Dirac [9] constraint analysis reveals that the model has, (as expected), two FCCs,

$$\chi \equiv \Pi_0 \approx 0 \;\; , \;\; G \equiv \partial_i \Pi_i - \frac{e}{n}\rho\alpha^n \approx 0, \tag{19}$$

where $\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}_i - \partial_i A_0$ and $\{\Pi_i(x), A_j(y)\} = -\delta_{ij}\delta(x - y)$, and the Hamiltonian is,

$$\mathcal{H} = \frac{\rho}{2}\bar{v}_i^2 + \frac{1}{2}\Pi_i^2 + \frac{1}{4}F_{ij}^2 + A_0(-\partial_i\Pi_i + \frac{e}{n}\rho\alpha^n), \tag{20}$$

with $\bar{v}_i = (\partial_i\theta + \alpha\partial_i\beta - e\frac{\alpha^n}{n}A_i)$. The constraint χ can be ignored since A_0 is just a multiplier field. The gauge invariant variables are $\rho,\ \alpha,\ \theta + \frac{(n-1)}{n}\alpha\beta,\ \Pi_i$ and $\bar{v}_i$.

We choose the rotationally invariant Coulomb gauge $\partial_i A_i \approx 0$ to gauge fix the Gauss law constraint G. The Dirac Brackets (DB) [9] are defined by

$$\{A(x), B(y)\}_{DB} = \{A(x), B(y)\} - \int (d^3z d^3w)\{A(x), \eta_a(z)\}\{\eta_a(z), \eta_b(w)\}^{-1}\{\eta_b(w), B(y)\},$$

which, in the present case produces the following algebra,

$$\{A_i(x), \Pi_j(y)\}_{DB} = (\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta(x-y),$$

$$\{\theta(x), \Pi_i(y)\}_{DB} = -e\frac{(1-n)}{n}\alpha^n(x)\frac{\partial_i^x}{\nabla^2}\delta(x-y) \quad , \quad \{\beta(x), \Pi_i(y)\}_{DB} = -e\alpha^{n-1}(x)\frac{\partial_i^x}{\nabla^2}\delta(x-y), \tag{21}$$

with rest of the brackets remaining unchanged. $\mathcal{H}$ in reduced phase space becomes,

$$\mathcal{H} = \frac{\rho}{2}\bar{v}_i^2 + \frac{1}{2}\Pi_i^2 + \frac{1}{2}(\partial_i A_j)^2 + \frac{e}{n}\rho\alpha^n. \tag{22}$$

(21) along with the other brackets lead to the gauge invariant algebra,

$$\{\bar{v}_i(x), \bar{v}_j(y)\}_{DB} = -\frac{\partial_i\bar{v}_j - \partial_j\bar{v}_i}{\rho}\delta(x-y) - e\frac{\alpha^n}{n\rho}F_{ij}\delta(x-y), \tag{23}$$

$$\{\bar{v}_i(x), \rho(y)\}_{DB} = \frac{\partial}{\partial x^i}\delta(x-y). \tag{24}$$

Hence the non-minimal MHD equations are

$$\dot{\bar{v}}_i = \bar{v}_j\partial_j\bar{v}_i - e\frac{\alpha^n}{n}(F_{ij}\bar{v}_j + \Pi_i) + \{\bar{v}_i, \int U\}, \tag{25}$$

$$\dot{\Pi}_i = \nabla^2 A_i + \frac{e}{n}\rho\alpha^n\bar{v}_i, \tag{26}$$

$$\dot{A}_i = \Pi_i - \partial_i\frac{\partial_j\Pi_j}{\nabla^2} = \Pi_i - \frac{e}{n}\partial_i\frac{\rho\alpha^n}{\nabla^2}. \tag{27}$$

But the non-local term in (27) drops out if time evolution of the magnetic field is considered. The conserved and gauge invariant current is $(\rho\alpha^n/n\ ,\ \rho\alpha^n\bar{v}_i/n)$. The above set of MHD equations is the main result of this section.

For the other conservation laws (11) and (12) the gauged Lagrangians are respectively,

$$\mathcal{L} = \rho(\dot{\theta} + \alpha\dot{\beta} - e\beta^n A_0) - \frac{\rho}{2}(\partial_i\theta + \alpha\partial_i\beta - e\beta^n A_i)^2 - U(\beta,\rho) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{28}$$

$$\mathcal{L} = \rho(\dot{\theta} + \alpha\dot{\beta} + e\frac{\alpha^n\beta^m}{m}A_0) - \frac{\rho}{2}(\partial_i\theta + \alpha\partial_i\beta + e\frac{\alpha^n\beta^m}{m}A_i)^2 - U(\alpha^n\beta^m,\rho) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{29}$$

The rest of the results are not reproduced here. This constitutes our analysis of the non-minimal and non-relativistic MHD.

We now discuss briefly the relativistic generalization of the above model where the above conserved currents survive in a modified form. In the relativistic generalization of the free theory [1], the Lagrangian is expressed as

$$\mathcal{L}_{rel} = j^\mu a_\mu - (j^\mu j_\mu)^{\frac{1}{2}},$$

$$a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta \ , \ j^\mu = (\rho, \ \rho v^i). \tag{30}$$

Notice that the symplectic structure does not change from the non-relativistic one. Expansion of the square root as

$$\rho(1 + v^i v_i)^{\frac{1}{2}} \approx \rho(1 + \frac{1}{2}v^i v_i + ...),$$

and dropping the uninteresting $\int \rho$ term [1], (since it can only influence the time evolution of θ by a constant translation), we can recover the non-relativistic Lagrangian in (6) with $U = 0$. The Hamiltonian now is modified to

$$\mathcal{H}_{rel} = \rho[v_i v_i + (1 - v_i v_i)^{\frac{1}{2}}], \tag{31}$$

which changes the equations of motion to the following:

$$\dot\alpha = L_i\partial_i\alpha \ \ ; \ \ \dot\beta = L_i\partial_i\beta \ \ ; \ \ \dot\theta = -L_i\alpha\partial_i\beta + [v_i v_i + (1 - v_i v_i)^{\frac{1}{2}}] \ \ ; \ \ \dot\rho = \partial_i(\rho L_i),$$

$$L_i = \frac{1}{\rho}\frac{\partial\mathcal{H}_{rel}}{\partial v_i} = v_i[2 - (1 - v_i v_i)^{-\frac{1}{2}}]. \tag{32}$$

Notice that in the lowest order, $L_i \approx v_i + O(v^3)$, the previous equations are recovered. Clearly the previous conservation laws will remain intact by replacing v_i by L_i in the expressions (10,11,12) for the currents.

To gauge the covariant system in (30), we introduce the interaction in the form,

$$\bar a_\mu = \alpha^{-n}(\partial_\mu\theta + \alpha\partial_\mu\beta) - \frac{e}{n}A_\mu \ \ ; \ \ \bar j^\mu = (\rho\alpha^n, \ \rho\alpha^n\bar v^i), \tag{33}$$

$$\bar{\mathcal{L}}_{rel} = \bar j^\mu \bar a_\mu - (\bar j^\mu \bar j_\mu)^{\frac{1}{2}}. \tag{34}$$

However, notice that in the non-relativistic limit, $\bar{\mathcal{L}}_{rel}$ reduces to $\mathcal{L}_{rel}$ in (30), but for the extra term $(-\rho\alpha^n)$. (Remember that a similar term $(-\rho)$ had to be dropped from the ungauged theory as well). But this term is not harmful in the sense that it is a perfectly valid interaction term $U(\rho, \alpha)$ that we have been considering.

Section II: Extended space quantization

This section deals with the Batalin-Tyutin (BT) [11] quantization of the fluid system. We start by embedding the system in a larger phase space having independent and commuting canonical pairs,

$$(\theta, \Pi_\theta \equiv \rho) \ \ ; \ \ (\alpha, \Pi_\alpha) \ \ ; \ \ (\beta, \Pi_\beta), \tag{35}$$

with $\{\theta(x), \Pi_\theta(y)\} = \delta(x - y)$ etc. Since we have introduced two extra variables in Π_α and Π_β, we also introduce two SCCs [8, 9]

$$\eta_1 \equiv \alpha\Pi_\theta - \Pi_\beta \ ; \ \ \eta_2 \equiv \Pi_\alpha \tag{36}$$

which reproduce (5) as DBs from the above canonical set and also keeps the degrees of freedom count same. Finally we bring in additional BT auxiliary variables [11] such that in the final BT-extended phase space the theory is converted into a gauge theory, having only the following commuting First Class Constraints (FCC) [9],

$$\tilde{\eta}_1 \equiv \eta_1 + \phi_1 \ ; \quad \tilde{\eta}_2 \equiv \eta_2 - \Pi_\theta \phi_2 \ , \ \{\tilde{\eta}_1, \tilde{\eta}_2\} = 0. \tag{37}$$

The BT fields obey $\{\phi_1(x), \phi_2(y)\} = \delta(x-y)$. To ensure that there are no further constraints, we need a Hamiltonian that commutes with the FCCs. The following variables,

$$\tilde{\theta} = \theta + \alpha\phi_2 \ , \ \tilde{\Pi}_\theta = \Pi_\theta \ , \ \tilde{\alpha} = \alpha + \frac{\phi_1}{\Pi_\theta} \ , \ \tilde{\Pi}_\alpha = \Pi_\alpha - \Pi_\theta \phi_2 \ , \ \tilde{\beta} = \beta - \phi_2 \ , \ \tilde{\Pi}_\beta = \Pi_\beta \ , \ \tilde{\phi}_i = 0. \tag{38}$$

are gauge invariant [11] in the sense that they commute with the FCCs. Hence *all* quantities written in terms of the redefined variables are gauge invariant in the extended space. In particular, the modified (free) Hamiltonian is

$$\tilde{\mathcal{H}} \mid_{free} = \frac{1}{2}(\tilde{\Pi}_\theta \tilde{v}_i \tilde{v}_i) = \frac{1}{2}\Pi_\theta [\partial_i(\theta + \alpha\phi_2) + (\alpha + \frac{\phi_1}{\Pi_\theta})\partial_i(\beta - \phi_2)]^2. \tag{39}$$

The remaining interaction terms in H, if present, will also be extended in a similar way. This Hamiltonian (39) together with the FCCs (37) and the canonical phase space is the gauge invariant system we were looking for. This constitutes the major result of the present section.

We note a fortuitous simplification in the extension structures (38). Unlike in other theories with *non-linear* SCCs [10], where some of the extensions turn out to be infinite sequences of higher order terms in ϕ_i-s, the present theory with non-linear SCCs (36), is free from this pathology.

To make contact with the physical system, the dimension of the BT extended phase space has to be reduced by additional gauge fixing constraints, (two in this case, $\tilde{\eta}_3$ and $\tilde{\eta}_4$, corresponding to two FCCs), with the only restriction that $\tilde{\eta}_a, \ a = 1,..,4$ constitute an SCC system that is $det \mid \{\tilde{\eta}_a, \tilde{\eta}_b\} \mid \neq 0$. A consistency check is to see that the original system is recovered in the so called unitary gauge, $\tilde{\eta}_3 \equiv \phi_1 \approx 0 \ \ , \ \ \tilde{\eta}_4 \equiv \phi_2 \approx 0$.

It might be convenient, (although not necessary), to consider the gauges of the form $\tilde{\eta}_3 \equiv \phi_1 - F \ , \ \tilde{\eta}_4 \equiv \phi_2 - G$, to remove the BT fields directly. F and G can contain the physical fields as well. For a particular gauge, one has to construct the corresponding DBs and compute the equations of motion using the DBs in reduced phase space, where the SCCs have been used strongly. Once again, the degrees of freedom count agrees with the original one. Consider the special class of gauge transformations: $\phi_1 = 0 \ , \ \phi_2 = constant$. These will *not* change the (v_i, ρ) algebra. Hence they can be identified as the conventional canonical transformations. Furthermore, additional constraints, such as incompressibility [13], can be included in this setup in the form $\rho = constant$, which under time persistence generates another constraint $\partial_i \tilde{v}_i$. This SCC pair leads to [13].

The constants of motion for the free theory are obviously the energy $\tilde{H}$, the momenta $\tilde{P}_i = \int(\rho\partial_i\theta + \Pi_\alpha\partial_i\alpha + \Pi_\beta\partial_i\beta + \phi_2\partial_i\phi_1)$, the angular momenta $\tilde{L}^{ij} = \int(r^i\tilde{\mathcal{P}}^j - r^j\tilde{\mathcal{P}}^i)$ and the boost generator $\tilde{B}^i = t\tilde{P}_i - \int(r_i\rho)$, effecting the transformation

$$\{\tilde{v}_i, u_j\tilde{B}_j\} = -t(u_j\partial_j)\tilde{v}_i + u_i \ , \ \{\rho, u_j\tilde{B}_j\} = -t(u_j\partial_j)\rho.$$

Obtaining the Lagrangian is indeed straightforward. The first order form is

$$\begin{aligned}\mathcal{L} &= \Pi_\theta\dot{\theta} + \Pi_\alpha\dot{\alpha} + \Pi_\beta\dot{\beta} + \phi_2\dot{\phi}_1 - \tilde{\mathcal{H}} - \lambda_1\tilde{\eta_1} - \lambda_2\tilde{\eta_2} \\ &\equiv \Pi_\theta\dot{\theta} + \phi_2\dot{\phi}_1 + \dot{\beta}(\alpha\Pi_\theta + \phi_1) + \dot{\alpha}\Pi_\theta\phi_2 - \tilde{\mathcal{H}} - \lambda_1\tilde{\eta_1} - \lambda_2\tilde{\eta_2},\end{aligned} \tag{40}$$

where λ_1 and λ_2 are multiplier fields and some of the variables have been removed using the equations of motion. At this stage, one can check explicitly that (40) is invariant under the following two independent sets of gauge transformations corresponding to the two FCCs,

$$\begin{aligned}\tilde{\eta_1} &\to \delta_1\Pi_\theta = 0, \delta_1\theta = -\alpha\psi_1, \delta_1\beta = \psi_1, \delta_1\alpha = 0, \delta_1\phi_1 = 0, \delta_1\phi_2 = \psi_1; \\ \tilde{\eta_2} &\to \delta_2\Pi_\theta = 0, \delta_2\theta = \phi_2\psi_2, \delta_2\beta = 0, \delta_2\alpha = -\psi_2, \delta_2\phi_1 = \Pi_\theta\psi_2, \delta_2\phi_2 = 0,\end{aligned} \tag{41}$$

where ψ_1 and ψ_2 are gauge transformation parameter functions. Naively taking the unitary gauge, *i.e.* $\phi_1 = \phi_2 = 0$, we can recover the Lagrangian posited in [1].

BT extension of the relativistic model is straightforward since the symplectic structure remains unchanged from the non-relativistic one. One only has to replace the original variables by their gauge invariant counterpart (38) in the covariant expressions (30) and (31).

It would be interesting to contrast the nature of (gauge) equivalence in the present theory with electrodynamics on one hand and Canonical Transfomation (CT) connecting inertial and non-inertial frames on the other hand. As an example in the former, MHD equations (25), (26) and (27), (taking magnetic field time evolution equation derived from (27)), are manifestly gauge invariant and the gauge transformation does not show up anywhere. However, in the latter, the same physical situation, if described in a non-inertial frame, (connected to an inertial one by time dependent CT), will require inertial or pseudo forces involving the transformation parameters. The BT extended gauge fluid theory is a generalization of the latter, since here, (unlike in CT), the Poisson Bracket structure can change under Gauge Transformations (GT). This and the appearance of interactions arising due to the choice of gauge, (analogous to pseudo interactions in CT), makes the equivalence non-trivial.

Also, presence of gauge invariance offers more freedom in the analysis of a theory and apparantly different models can become identified as gauge equivalent ones, so that results obtained in one model can be carried to the other one. For instance, generally one considers the fluid system, in physical situation, as being subjected to a pressure term $(\frac{\partial_i p(x)}{\rho})$ and a constant force such as gravity. By a suitable choice of (translation symmetry breaking) gauge, one can generate these terms in the Euler equation. However, even in the linear approximation, there will be additional terms in the Euler equation besides the above ones and the continuity equation will also be modified. One can say that this set is gauge equivalent to the free theory since the latter is reproduced in the unitary gauge. Indeed. one can trade one type of interaction and source with another one by exploiting the BT gauge equivalence and one set might be better suited to simulate experimentally or analyse theoretically.

Section IV: Conclusions and future prospects

At present, study of the Hamiltonian fluid dynamics is an active area. The novel $U(1)$ gauge interaction that we have proposed here, along with an appropriate potential term U, can lead to interesting fluid profiles due to the inherent non-linearity present in the fluid system and the gauging mechanism. One can also gauge more than one invariance simultaniously.

A non-abelian Clebsch parametrization has been proposed in [14]. It would be very interesting to see if the non-abelian counterpart also supports this non-minimal gauging prescription.

Furthermore, gauge invariant interactions of the form $\epsilon_{\mu\nu\sigma\lambda}F^{\mu\nu}\theta\partial^{\sigma}\alpha\partial^{\lambda}\beta$ can also be introduced in the action which will alter the symplectic structure, that can be studied perturbatively. (Unfortunately, the topological current in the above term, $\epsilon_{\mu\nu\sigma\lambda}\partial^{\nu}\theta\partial^{\sigma}\alpha\partial^{\lambda}\beta$ does not carry a topological charge, *i.e.* helicity, mentioned before.)

The Batalin-Tyutin construction is primarily aimed at providing a canonical framework for quantizing the fluid system. It will indeed be interesting to study the quantized version of the above model. These are some of the areas that we intend to look at in near future.

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